

# RENORMALIZATION GROUP APPROACH TO FIELD THEORY AT FINITE TEMPERATURE

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## ABSTRACT

Scalar field theory at finite temperature is investigated via an improved renormalization group prescription which provides an effective resummation over all possible non-overlapping higher loop graphs. Explicit analyses for the  $\lambda\phi^4$  theory are performed in  $d = 4$  Euclidean space for both low and high temperature limits. We generate a set of coupled equations for the mass parameter and the coupling constant from the renormalization group flow equation. Dimensional reduction and symmetry restoration are also explored with our improved approach.

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# I. INTRODUCTION

In recent years, intensive efforts have been devoted to quantum field theory at finite temperature, a subject with wide applications in areas such as the evolution of the early universe and its cosmological consequences [1], the deconfinement phenomena and formation of the quark-gluon plasma [2], and the critical behavior of condensed matter systems near the phase transition. To investigate these issues, one generally utilizes the finite-temperature effective potential approach in the spirit of the perturbative loop expansion. However, in the high temperature limit, perturbation theory becomes unreliable [3] since the presence of infrared (IR) divergences may destroy the correspondence between the expansions of loops and the coupling constant. Certain higher loop contributions such as the “daisy” and “superdaisy” diagrams that contribute to the same order in the coupling constant must also be incorporated [4] for computing the critical transition temperature  $T_c$  and determining the nature of the phase transition. In gauge theories, it has been shown that the “hot thermal loops” need to be resummed in order to obtain a gauge independent gluon damping rate [5].

However, since resumming the multi-loop contributions to arbitrary orders proves to be nontrivial, various methods have been proposed to carry out the resummation: In [4], the use of a gap equation was first discussed; in [6] a renormalization group (RG) with  $T$ , the temperature, as the flow parameter was used; and in [7] a self-consistent Hartree-Fock formalism is presented. An “environmentally friendly” renormalization prescription for interpolating effective finite-temperature theories in different regimes has also been proposed [8]. Although all of these techniques yield the same finite-temperature effective propagators in the leading order, they differ in the subleading correction which can affect the nature of the phase transition.

In this paper, the methodology adopted for investigating the scalar field theory at finite temperature is based on the use of RG constructed from the Wilson-Kadanoff [9] blocking transformation in the Euclidean formalism. Unlike [6], our RG arises from the arbitrariness of the internal blocking scale  $k$ , and not the external temperature parameter  $T$ . The formulation not only takes into consideration the dominant higher loop diagrams without resorting to the complicated analytical order by order resummation, it also characterizes the flow pattern of the theory for arbitrary  $T$  as well as the momentum scale  $k$  which is chosen to be zero in the methods described above. Our approach is analogous to the series of works by Tetradis and Wetterich [10]. However, instead of using a smooth momentum smearing function which leads to an integro-differential RG equation, our sharp momentum cut-off [11] yields a RG flow equation which takes on the form of non-linear partial differential equation. Besides the advantage in performing numerical computation, it also offers a more lucid physical interpretations.

Consider the following scalar lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + V(\phi). \quad (1.1)$$

At finite temperature, the RG improved blocked potential  $U_{\beta,k}(\Phi)$  associated with the blocked field  $\Phi(x)$  is characterized by the following differential RG equation [11] :

$$k \frac{\partial U_{\beta,k}}{\partial k} = -\frac{k^3}{4\pi^2} \sqrt{k^2 + U''_{\beta,k}} - T \frac{k^3}{2\pi^2} \ln \left[ 1 - e^{-\beta \sqrt{k^2 + U''_{\beta,k}}} \right]. \quad (1.2)$$

The above equation is obtained by first computing the finite-temperature blocked potential to the one-loop order followed by a RG improvement to take into account all possible non-overlapping daisy, superdaisy and higher-loop diagrams which strongly modify theory in the high  $T$  and small  $k$  limits. We also see that in this RG construction the contributions of a particular mode which has been integrated out are naturally retained for the integration of the next, and therefore the interactions among the modes are properly taken into account. In addition, during the course of mode elimination, irrelevant operators defined with respect to the ultraviolet (UV) fixed point continue to be generated and their effects incorporated as well by (1.2) [12]. Furthermore, our flow equation also offers insights into the existence of high temperature dimensional reduction as well as the restoration of symmetry accompanied by the disappearance of imaginary contribution to  $U_{\beta,k}(\Phi)$ , as we shall see later. To describe the full theory, however, one also needs to consider the effects of wavefunction renormalization constant  $Z_{\beta,k}(\Phi)$  as well as the higher-order derivative terms in the blocked lagrangian. Nevertheless, the use of (1.2) is justified in the IR limit of the four-dimensional theories since the other contributions are only of higher order.

The organization of the paper is as follows: In Sec. II the formalism of the blocking transformation at finite temperature is briefly reviewed using the scalar  $\lambda\phi^4$  theory as illustration. The finite temperature RG flow equation (1.2) with underlying  $O(3)$  symmetry is constructed and compared with the simple one-loop independent mode approximation (IMA). In Sec. III we concentrate on the low  $T$  regime where thermal effects are negligible and compare the  $O(3)$  thermal blocked potential  $U_{\beta,k}(\Phi)$  with the  $O(4)$  symmetric  $U_k(\Phi)$  at  $T = 0$ . The high  $T$  limit of the theory is investigated in Section IV. The criterion for dimensional reduction is explicitly deduced from (1.2). In addition, the behaviors of the scale-dependent thermal mass and coupling constant  $\mu_{\beta,k}^2$  and  $\lambda_{\beta,k}$ , are studied with a set of coupled equations. It is shown that  $\lambda_{\beta,k}$  decreases with increasing  $T$  and approaches a constant in this regime. The complementary relationship between the internal scale  $k$  and the external parameter  $T$  is also demonstrated. The phenomenon of spontaneous symmetry breaking at  $T = 0$  and symmetry restoration above  $T_c$  is discussed in Sec. VI. The phase boundary is determined from (1.2) by requiring the imaginary part of  $U_{\beta,k}(\Phi)$  to vanish at  $T_c$ . The value obtained in this manner is compared with that obtained in the one-loop approximation without the imaginary sector. Sec. VI is reserved for summary and discussions. We collect in the Appendix the details of extracting the leading order thermal contributions from the integrals encountered in Sec. IV.

## II. FINITE-TEMPERATURE RENORMALIZATION GROUP

In this section, we briefly review the finite temperature RG formalism used in investigating the thermal behavior of the theory. The concept of the RG is based on the notion that in certain physical processes only a particular range of modes in the momentum decomposition of the field  $\phi(x)$  will be relevant, and it is often desirable to eliminate the irrelevant modes to which the physics is insensitive. The reduction of degrees of freedom is then compensated by a readjustment of the parameters in the coupling constant space of the lagrangian. Therefore, instead of using the original field variable

$$\phi(x) = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int_{\mathbf{p}} e^{-i(\omega_n \tau - \mathbf{p} \cdot \mathbf{x})} \phi(\omega_n, \mathbf{p}), \quad \int_{\mathbf{p}} = \int \frac{d^3 \mathbf{p}}{(2\pi)^3}, \quad (2.1)$$

where  $\beta^{-1} = T$  and  $\omega_n = 2\pi n/\beta$  denotes the Matsubara frequency, we define the coarse-grained blocked field as [11]:

$$\Phi(\mathbf{x}) = T \int_0^\beta dy^0 \int d^3\mathbf{y} \rho_k(\mathbf{x} - \mathbf{y}) \phi(y) \quad (2.2)$$

via an  $O(3)$  invariant smearing function  $\rho_k(\mathbf{x})$ . Since the low energy physics is unaffected by the modes above the “blocking scale”  $k$ , we shall for simplicity choose

$$\rho_k(\mathbf{x}) = \int_{|\mathbf{p}| < k} \frac{d^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}}, \quad (2.3)$$

or  $\rho_k(\mathbf{p}) = \Theta(k - |\mathbf{p}|)$ . In other words,  $k$  acts as an upper cut-off for the modes which are to be retained. Notice that in this formulation, the  $\delta$ -function associated with the imaginary-time variable  $\tau$  is given by

$$\delta(\tau_x - \tau_y) = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} e^{-i\omega_n(\tau_x - \tau_y)}. \quad (2.4)$$

Since the wavefunction renormalization constant  $Z_k(\Phi)$  yields only a minute correction to the anomalous dimension, we shall for simplicity set  $Z_k(\Phi)$  to be unity. In addition, by neglecting the higher order field derivative terms, one may simply choose the static limit  $\Phi(\mathbf{x}) = \Phi = \text{const.}$  for computing  $U_{\beta,k}(\Phi)$ . Upon a simple Gaussian integration, the one-loop contribution to the finite-temperature blocked potential  $\tilde{U}_{\beta,k}(\Phi)$  becomes

$$\begin{aligned} \tilde{U}_{\beta,k}^{(1)}(\Phi) &= \frac{1}{2\beta} \sum_{n=-\infty}^{\infty} \int'_{\mathbf{p}} \ln[\omega_n^2 + p^2 + V''(\Phi)] \\ &= \frac{1}{2\beta} \int'_{\mathbf{p}} \left\{ \beta \sqrt{p^2 + V''(\Phi)} + 2 \ln[1 - e^{-\beta \sqrt{p^2 + V''(\Phi)}}] \right\}, \end{aligned} \quad (2.5)$$

where the prime on the  $\mathbf{p}$  integral indicates an integration subject to the constraint  $k \leq \mathbf{p} \leq \Lambda$ , with  $\Lambda$  chosen as the UV regulator for the three-momentum integration. For  $V(\Phi) = \mu^2\Phi^2/2 + \lambda\Phi^4/4!$ , by demanding the renormalized parameters to satisfy

$$\begin{cases} \tilde{\mu}_R^2 = \frac{\partial^2 \tilde{U}_{\beta,k}}{\partial \Phi^2} \Big|_{\Phi=\beta^{-1}=k=0} \\ \tilde{\lambda}_R = \frac{\partial^4 \tilde{U}_{\beta,k}}{\partial \Phi^4} \Big|_{\Phi=\beta^{-1}=k=0}, \end{cases} \quad (2.6)$$

the resulting  $\tilde{U}_{\beta,k}(\Phi)$  takes on the form:

$$\begin{aligned} \tilde{U}_{\beta,k}(\Phi) &= \frac{\tilde{\mu}_R^2}{2} \Phi^2 \left(1 - \frac{\tilde{\lambda}_R}{64\pi^2}\right) + \frac{\tilde{\lambda}_R}{4!} \Phi^4 \left(1 - \frac{9\tilde{\lambda}_R}{64\pi^2}\right) \\ &+ \frac{1}{32\pi^2} \left\{ -k \left(2k^2 + \tilde{\mu}_R^2 + \frac{1}{2}\tilde{\lambda}_R\Phi^2\right) \left(k^2 + \tilde{\mu}_R^2 + \frac{1}{2}\tilde{\lambda}_R\Phi^2\right)^{1/2} \right. \\ &+ \left. \left(\tilde{\mu}_R^2 + \frac{1}{2}\tilde{\lambda}_R\Phi^2\right)^2 \ln \left[ \frac{k + \sqrt{k^2 + \tilde{\mu}_R^2 + \tilde{\lambda}_R\Phi^2/2}}{\tilde{\mu}_R} \right] \right\} \\ &+ \frac{1}{2\pi^2\beta} \int_k^\Lambda dp p^2 \ln[1 - e^{-\beta \sqrt{p^2 + \tilde{\mu}_R^2 + \tilde{\lambda}_R\Phi^2/2}}], \end{aligned} \quad (2.7)$$

which in the limits of vanishing  $\beta^{-1}$  and  $k$ , reproduces the usual effective potential [13]. Therefore, one sees that consistency with the  $O(4)$  invariant theory in the  $T = 0$  limit requires a complete summation over the Matsubara frequencies  $\omega_n = 2\pi n/\beta$  for all  $n$ , i.e.,  $\rho_k(\mathbf{p})$  must be independent of  $\omega_n$ . It is also evident from (2.6) that temperature-independent subtractions alone are sufficient to remove the cut-off dependence.

Differentiating (2.5) with respect to the arbitrary scale  $k$  leads to

$$k \frac{\partial \tilde{U}_{\beta,k}}{\partial k} = -\frac{k^3}{4\pi^2} \sqrt{k^2 + V''(\Phi)} - T \frac{k^3}{2\pi^2} \ln \left[ 1 - e^{-\beta \sqrt{k^2 + V''(\Phi)}} \right]. \quad (2.8)$$

This RG equation is obtained in a manner in which each individual mode is integrated out independently by neglecting the systematic feedbacks from the high modes to the lower ones during the elimination. The parameters for this independent-mode approximation (IMA) scheme are denoted by a tilde.

On the other hand, instead of integrating out all the modes from  $\Lambda$  to  $k$  all at once, one may first divide the integration volume into a large number of thin shells of small thickness  $\Delta k$ . By lowering the cut-off infinitesimally from  $\Lambda \rightarrow \Lambda - \Delta k$  until  $\Lambda = k$  is reached, we arrive at the following RG equation governing the flow pattern of the *improved* finite-temperature blocked potential  $U_{\beta,k}(\Phi)$ :

$$k \frac{\partial U_{\beta,k}}{\partial k} = -\frac{k^3}{4\pi^2} \sqrt{k^2 + U''_{\beta,k}} - T \frac{k^3}{2\pi^2} \ln \left[ 1 - e^{-\beta \sqrt{k^2 + U''_{\beta,k}}} \right]. \quad (2.9)$$

This non-linear partial differential equation establishes a smooth connection between the small- and large-distance physics at finite temperature. Moreover, (2.9) systematically incorporates the contribution of a particular mode for the elimination of the next. In fact, (2.9) takes into account all possible non-overlapping daisy, superdaisy and higher-loop diagrams which significantly modify the theory at high  $T$  and small  $k$  limits. The summation over the multi-loop graphs from the basic one-loop structure is depicted in Fig. 1. In other words, what we have here is a “physical” one-loop diagram characterized by the dressed vertices. One may certainly improve (2.9) by including higher “physical” loop contributions. However, we argue that since each loop integration is multiplied by a factor  $\kappa = \Delta k/k$ , a term of order  $m$  in loops will be suppressed by  $\kappa^m$  in the limit  $\Delta k \rightarrow 0$ . Therefore, the use of (2.9) is justified. What we have neglected in (2.9) are the overlapping higher loop diagrams. However, as shown in [4], if one considers the  $N$ -component theory and takes the limit  $N \rightarrow \infty$ , these overlapping graphs will be suppressed by an extra factor  $N$  compared with the corresponding non-overlapping contributions at the same loop order. The effects of these overlapping graphs for the one-component case is currently being explored.

Eq. (2.9) implies the following RG flow equation for the effective scale-dependent thermal parameters  $\mu_{\beta,k}^2 = U''_{\beta,k}(0)$ ,  $\lambda_{\beta,k} = U^{(4)}_{\beta,k}(0)$  and  $g_{\beta,k} = U^{(6)}_{\beta,k}(0)$ :

$$k \frac{\partial \mu_{\beta,k}^2}{\partial k} = -\frac{\lambda_{\beta,k}}{8\pi^2} \frac{k^3}{\sqrt{k^2 + \mu_{\beta,k}^2}} \coth \left( \frac{\beta \sqrt{k^2 + \mu_{\beta,k}^2}}{2} \right), \quad (2.10)$$

and

$$\begin{aligned}
k \frac{\partial \lambda_{\beta,k}}{\partial k} = & \frac{3\lambda_{\beta,k}^2}{16\pi^2} \frac{k^3}{k^2 + \mu_{\beta,k}^2} \left\{ \frac{1}{\sqrt{k^2 + \mu_{\beta,k}^2}} \coth\left(\frac{\beta\sqrt{k^2 + \mu_{\beta,k}^2}}{2}\right) + \frac{e^{\beta\sqrt{k^2 + \mu_{\beta,k}^2}}}{T(e^{\beta\sqrt{k^2 + \mu_{\beta,k}^2}} - 1)^2} \right\} \\
& - \frac{3g_{\beta,k}}{8\pi^2} \frac{k^3}{\sqrt{k^2 + \mu_{\beta,k}^2}} \left[ 1 + \frac{2}{3(e^{\beta\sqrt{k^2 + \mu_{\beta,k}^2}} - 1)} \right].
\end{aligned} \tag{2.11}$$

Notice the contribution to the flow of  $\lambda_{\beta,k}$  from the sixth-order coupling constant  $g_{\beta,k}$ . The presence of such higher order corrections is the natural consequence of blocking transformation through which the number of effective modes is reduced at the expense of generating a more complicated effective action.

While  $\tilde{U}_{\beta,k}(\Phi)$  associated with the IMA is solved analytically in (2.7) apart from the integral containing thermal effects, the RG improved  $U_{\beta,k}(\Phi)$  can only be obtained by solving (2.9) numerically. In Figs. 2 and 3, the evolution of  $U_{\beta,k}(\Phi)$  and  $\tilde{U}_{\beta,k}(\Phi)$  for  $T = 0$  and  $T \neq 0$  are traced. We notice that the two potentials agree reasonably well for  $k$  near the cut-off  $\Lambda$  and begin to deviate as  $k$  is lowered. In the deep IR limit where  $k = 0$ , appreciable difference between  $U_{\beta,k}(\Phi)$  and  $\tilde{U}_{\beta,k}(\Phi)$  is seen. Such a difference can be understood by noting that

$$\begin{aligned}
k \frac{\partial}{\partial k} (U_{\beta,k}(\Phi) - \tilde{U}_{\beta,k}(\Phi)) = & -\frac{k^3}{4\pi^2} \left\{ \sqrt{k^2 + U''_{\beta,k}(\Phi)} - \sqrt{k^2 + V''(\Phi)} \right\} \\
& - \frac{Tk^3}{2\pi^2} \ln \left[ \frac{1 - e^{-\beta\sqrt{k^2 + U''_{\beta,k}(\Phi)}}}{1 - e^{-\beta\sqrt{k^2 + V''(\Phi)}}} \right] \\
= & -\frac{k^3}{4\pi^2} \sqrt{k^2 + V''(\Phi)} \left\{ \left[ 1 + \frac{U''_{\beta,k}(\Phi) - V''(\Phi)}{k^2 + V''(\Phi)} \right]^{1/2} - 1 \right\} \\
& - \frac{Tk^3}{2\pi^2} \ln \left[ 1 + \frac{\exp\{-\beta(\sqrt{k^2 + V''(\Phi)} - \sqrt{k^2 + U''_{\beta,k}(\Phi)})\}}{1 - \exp\{-\beta\sqrt{k^2 + V''(\Phi)}\}} \right] \\
= & -\frac{k^3}{8\pi^2} \frac{(U''_{\beta,k}(\Phi) - V''(\Phi))}{\sqrt{k^2 + V''(\Phi)}} \left[ 1 - \frac{1}{4} \left( \frac{U''_{\beta,k}(\Phi) - V''(\Phi)}{k^2 + V''(\Phi)} \right) \right] \\
& \times \left\{ 1 - \frac{2T}{\sqrt{k^2 + V''(\Phi)}} (e^{\beta\sqrt{k^2 + V''(\Phi)}} - 1)^{-1} \right\} + \dots
\end{aligned} \tag{2.12}$$

Thus we see that higher loop corrections will continue to pile up as  $k$  is lowered or  $T$  is raised. By comparing Fig.2 and Fig. 3, we see that the inclusion of the thermal effects improves the agreement between the RG and the IMA results. In Sec. IV we shall see that the high  $T$  behavior of the theory is strongly modified by the higher loop contributions.

In Fig. 4 the evolutions of  $\mu_{\beta,k}^2$  and  $\lambda_{\beta,k}$  with  $k$  for three different values of  $T$  are depicted. While  $\mu_{\beta,k}^2$  increases with  $T$ ,  $\lambda_{\beta,k}$  is shown to decrease with rising  $T$ . On the

other hand, when the role of  $k$  is considered, its influences on  $\mu_{\beta,k}^2$  and  $\lambda_{\beta,k}$  are completely opposite to that of  $T$ . One therefore observes an interesting competition between the external parameter  $T$  and the internal momentum scale parameter  $k$  from which RG is defined. By external parameter we mean a “physical” quantity which has direct physical impact on the system and can be measured experimentally. On the other hand, an internal scale is a “fictitious” scale which we choose to characterize the theory. In this case,  $k$  is chosen to provide a separation between the high and the low modes. However, the precise definition of “high” and “low” modes is dependent on the energy regime under investigation. A more complete discussion concerning the behaviors of the thermal parameters for different  $k$  will appear in the later sections.

### III. LOW TEMPERATURE LIMIT

We first consider the low temperature limit  $T \ll \sqrt{k^2 + U''_{\beta,k}}$  where, upon dropping the subscript  $\beta$ , (2.9) is reduced to

$$k \frac{\partial U_k}{\partial k} = -\frac{k^3}{4\pi^2} \sqrt{k^2 + U''_k}. \quad (3.1)$$

This “zero” temperature limit can be compared with the RG flow equation associated with  $\hat{U}_{\hat{k}}(\Phi)$ , the blocked potential for  $d = 4$  derived in an  $O(4)$  symmetric manner [14]:

$$\hat{k} \partial_{\hat{k}} \hat{U}_k(\Phi) = -\frac{\hat{k}^4}{16\pi^2} \ln \left( \frac{\hat{k}^2 + \hat{U}''_{\hat{k}}(\Phi)}{\hat{k}^2 + \hat{U}''_{\hat{k}}(0)} \right) \quad (3.2)$$

Note that quantities derived from the  $O(4)$  symmetric  $\hat{U}_{\hat{k}}(\Phi)$  shall be distinguished with a caret, and double carets for the corresponding IMA scheme.

Since the underlying symmetry of the finite temperature RG equation (2.9) is  $O(3)$ , it is instructive to compare its low  $T$  flow patterns with (3.2) for both RG improved and IMA prescriptions. Eqs. (3.1), (3.2) along with (2.8) lead to the following RG coefficient functions for the running mass parameter and the running coupling constants:

$$\left\{ \begin{array}{ll} \tilde{\beta}_2 = k \frac{\partial \tilde{\mu}_k^2}{\partial k} = -\frac{\tilde{\lambda}_R}{8\pi^2} \frac{k^3}{\sqrt{k^2 + \tilde{\mu}_R^2}} & [O(3), \text{ IMA}] \\ \beta_2 = k \frac{\partial \mu_k^2}{\partial k} = -\frac{\lambda_k}{8\pi^2} \frac{k^3}{\sqrt{k^2 + \mu_k^2}} & [O(3), \text{ RG}] \\ \hat{\beta}_2 = \hat{k} \frac{\partial \hat{\mu}_{\hat{k}}^2}{\partial \hat{k}} = -\frac{\hat{\lambda}_R}{16\pi^2} \frac{\hat{k}^4}{\hat{k}^2 + \hat{\mu}_R^2} & [O(4), \text{ IMA}] \\ \hat{\beta}_2 = \hat{k} \frac{\partial \hat{\mu}_{\hat{k}}^2}{\partial \hat{k}} = -\frac{\hat{\lambda}_{\hat{k}}}{16\pi^2} \frac{\hat{k}^4}{\hat{k}^2 + \hat{\mu}_{\hat{k}}^2} & [O(4), \text{ RG}] \end{array} \right. \quad (3.3)$$

$$\left\{ \begin{array}{ll} \tilde{\beta}_4 = k \frac{\partial \tilde{\lambda}_k}{\partial k} = \frac{3\tilde{\lambda}_R^2}{16\pi^2} \frac{k^3}{(k^2 + \tilde{\mu}_R^2)^{3/2}} & [\text{O}(3), \text{ IMA}] \\ \beta_4 = k \frac{\partial \lambda_k}{\partial k} = \frac{3\lambda_k^2}{16\pi^2} \frac{k^3}{(k^2 + \mu_k^2)^{3/2}} - \frac{g_k}{8\pi^2} \frac{k^3}{\sqrt{k^2 + \mu_k^2}} & [\text{O}(3), \text{ RG}] \\ \hat{\beta}_4 = \hat{k} \frac{\partial \hat{\lambda}_{\hat{k}}}{\partial \hat{k}} = \frac{3\hat{\lambda}_R^2}{16\pi^2} \frac{\hat{k}^4}{(\hat{k}^2 + \hat{\mu}_R^2)^2} & [\text{O}(4), \text{ IMA}] \\ \hat{\beta}_4 = \hat{k} \frac{\partial \hat{\lambda}_{\hat{k}}}{\partial \hat{k}} = \frac{3\hat{\lambda}_{\hat{k}}^2}{16\pi^2} \frac{\hat{k}^4}{(\hat{k}^2 + \hat{\mu}_{\hat{k}}^2)^2} - \frac{\hat{g}_{\hat{k}}}{16\pi^2} \frac{\hat{k}^4}{\hat{k}^2 + \hat{\mu}_{\hat{k}}^2} & [\text{O}(4), \text{ RG}]. \end{array} \right. \quad (3.4)$$

Notice that in the RG improved  $O(3)$  and  $O(4)$  schemes for  $\lambda_k$ , the influences from  $g_k$  is incorporated. For  $k^2 \gg \mu_k^2$  and  $\hat{k}^2 \gg \hat{\mu}_k^2$ , one finds agreement in the leading order behavior of  $\beta_4$  and  $\hat{\beta}_4$  for the running coupling constant, as can be seen from (3.4). However, for the running mass parameter,  $\beta_2$  and  $\hat{\beta}_2$  differ by a factor of two in this limit. Such a discrepancy can easily be understood by noting the difference in the symmetries of the underlying RG constructs, namely,  $O(3)$  for the former and  $O(4)$  for the latter. The manner in which the original bare theory is approached can be elucidated by examining the simple one-loop structure of the bare mass parameter:

$$\begin{aligned} \mu_B^2 &= \tilde{\mu}_R^2 - \frac{\tilde{\lambda}_R}{32\pi^2} \left[ 2\Lambda^2 - \tilde{\mu}_R^2 \ln\left(\frac{2\Lambda^2}{\tilde{\mu}_R^2}\right) + \tilde{\mu}_R^2 (1 - \ln 2) \right] & : O(3) \\ &= \tilde{\mu}_R^2 - \frac{\tilde{\lambda}_R}{32\pi^2} \left[ \hat{\Lambda}^2 - \tilde{\mu}_R^2 \ln\left(\frac{\hat{\Lambda}^2}{\tilde{\mu}_R^2}\right) \right] & : O(4), \end{aligned} \quad (3.5)$$

where  $\Lambda$  and  $\hat{\Lambda}$  are, respectively, the three- and four-momentum ultraviolet cut-offs. Therefore, in the low-temperature limit with  $k^2 \gg \mu_{\beta,k}^2$ ,  $U_{\beta,k}(\Phi)$  can be transformed to  $\hat{U}_{\hat{k}}(\Phi)$  with the approximate scaling relation  $k \rightarrow \hat{k}/\sqrt{2}$ . The reason for the existence of scaling can be explained by noting that the one-loop zero temperature contribution to  $\tilde{U}_{\beta,k}(\Phi)$  from (2.5) can be rewritten at  $k = 0$  as:

$$\tilde{U}_{k=0}^{(1)}(\Phi) = \frac{1}{2} \int_0^\Lambda \frac{d^3 \mathbf{p}}{(2\pi)^3} \sqrt{\mathbf{p}^2 + V''(\Phi)} = \frac{1}{2} \int_k^\infty \frac{d^3 \mathbf{p}}{(2\pi)^3} \int_{-\infty}^\infty \frac{dp_0}{2\pi} \ln[p_0^2 + \mathbf{p}^2 + V''(\Phi)], \quad (3.6)$$

where we have used

$$\int_{-\infty}^\infty \frac{dp_0}{2\pi} \ln[p_0^2 + E^2] = E, \quad (3.7)$$

which holds up to an  $E$ -independent constant. With the coordinate transformation

$$\begin{cases} p_0 = p \sin \theta \\ \mathbf{p} = p \cos \theta \end{cases} \quad (3.8)$$

and imposing the same cut-off regulator  $\Lambda$  for  $p_0$  such that  $0 \leq p^2 \leq 2\Lambda^2$ , the above expression becomes:

$$\begin{aligned}
\tilde{U}_{k=0}^{(1)}(\Phi) &= \frac{1}{4\pi^2} \int_0^\Lambda d\mathbf{p} \mathbf{p}^2 \int_0^\Lambda dp_0 \ln[p_0^2 + \mathbf{p}^2 + V''(\Phi)] \\
&= \frac{1}{4\pi^2} \int_0^{\pi/2} d\theta \cos^2\theta \int_0^{\sqrt{2}\Lambda} dp p^3 \ln[p^2 + V''(\Phi)] \\
&\quad \frac{1}{16\pi^2} \int_0^{\sqrt{2}\Lambda} dp p^3 \ln[p^2 + V''(\Phi)] \\
&= \frac{1}{2} \int_0^{\sqrt{2}\Lambda} \frac{d^4 p}{(2\pi)^4} \ln[p^2 + V''(\Phi)].
\end{aligned} \tag{3.9}$$

Therefore, we see that in the UV regime where  $k \approx \Lambda$ , with the rescaling  $\Lambda = \tilde{\Lambda}/\sqrt{2} \rightarrow \infty$ , the results derived from  $O(3)$  and  $O(4)$  RG schemes agree with each other.

On the other hand, in the IR limit where  $k^2 \ll \mu_k^2$  and  $\hat{k}^2 \ll \hat{\mu}_k^2$ , we have

$$\left\{ \begin{array}{ll} \tilde{\beta}_2 = -\frac{\tilde{\lambda}_R k^3}{16\pi^2 \tilde{\mu}_R} \left(1 - \frac{1}{2} \frac{k^2}{\tilde{\mu}_R^2}\right) + \dots & [O(3), \text{ IMA}] \\ \beta_2 = -\frac{\lambda_k k^3}{8\pi^2 \mu_k} \left(1 - \frac{1}{2} \frac{k^2}{\mu_k^2}\right) + \dots & [O(3), \text{ RG}] \\ \hat{\beta}_2 = -\frac{\hat{\lambda}_R \hat{k}^4}{16\pi^2 \hat{\mu}_R^2} \left(1 - \frac{\hat{k}^2}{\hat{\mu}_R^2}\right) + \dots & [O(4), \text{ IMA}] \\ \hat{\beta}_2 = -\frac{\hat{\lambda}_k \hat{k}^4}{16\pi^2 \hat{\mu}_k^2} \left(1 - \frac{\hat{k}^2}{\hat{\mu}_k^2}\right) + \dots & [O(4), \text{ RG}], \end{array} \right. \tag{3.10}$$

and

$$\left\{ \begin{array}{ll} \tilde{\beta}_4 = \frac{3\tilde{\lambda}_R^2 k^3}{16\pi^2 \tilde{\mu}_R^3} \left(1 - \frac{3k^2}{2\tilde{\mu}_R^2}\right) + \dots & [O(3), \text{ IMA}] \\ \beta_4 = \frac{3\lambda_k^2 k^3}{16\pi^2 \mu_k^3} \left(1 - \frac{3k^2}{2\mu_k^2}\right) - \frac{g_k k^3}{8\pi^2 \mu_k} \left(1 - \frac{k^2}{2\mu_k^2}\right) + \dots & [O(3), \text{ RG}] \\ \hat{\beta}_4 = \frac{3\hat{\lambda}_R^2 \hat{k}^4}{16\pi^2 \hat{\mu}_R^2} \left(1 - \frac{2\hat{k}^2}{\hat{\mu}_R^2}\right) + \dots & [O(3), \text{ IMA}] \\ \hat{\beta}_4 = \frac{3\hat{\lambda}_k^2 \hat{k}^4}{16\pi^2 \hat{\mu}_k^2} \left(1 - \frac{2\hat{k}^2}{\hat{\mu}_k^2}\right) - \frac{\hat{g}_k k^4}{16\pi^2 \hat{\mu}_k^2} \left(1 - \frac{\hat{k}^2}{\hat{\mu}_k^2}\right) + \dots & [O(4), \text{ RG}]. \end{array} \right. \tag{3.11}$$

Comparing (3.10) and (3.11), one sees that the running parameters increase more rapidly for the  $O(3)$  symmetric case near  $k \approx 0$  by a factor of  $\mu_k/k$  in the IR regime, and the scaling relation between  $k$  and  $\hat{k}$  for transforming  $U_{\beta,k}(\Phi)$  and  $\hat{U}_{\hat{k}}(\Phi)$  in the UV regime is clearly lost. This is due to the fact that (3.7) no longer holds for finite  $k$  and that a

transformation similar to that in (3.9) for finite  $k$  ceases to exist. The numerical results for the flow equations in (3.3) and (3.4) are illustrated in Figs. 5 and 6. For the mass parameters shown in Fig. 5, the flow begins at  $\Lambda = \hat{\Lambda}/\sqrt{2}$  where all schemes take on the same negative bare mass parameter  $\mu_B^2$ . We have chosen  $\mu_B^2 < 0$  although in principle it can also be positive. However, the physics we are interested lies in the IR regime which is very far from the cut-off. Therefore, the low energy physics is influenced by the sign of the renormalized mass parameter but not of  $\mu_B^2$ . The scaling relation between the  $O(3)$  and the  $O(4)$  schemes in the large  $k$  limit is seen by noting that the two curves are approximately parallel to each other initially. One also observes that both RG improved prescriptions yield the same mass parameter at  $k = 0$ , independent of the underlying symmetry of the RG. Such a behavior is also observed within the IMA schemes. This symmetry-independent behavior is again due to the transformation (3.9) which leads to identical one-loop effective potential. The same argument can also be made for the RG improved prescriptions by simply replacing  $V''(\Phi)$  in (3.9) by the corresponding  $U''_{\beta,k}(\Phi)$ . This also explains the necessity of having a more rapid convergence in the deep IR limit in order that both RG schemes yield the same improved parameters.

Another notable feature arises from comparing IMA with the RG approach, where we see that by choosing an initial condition  $\mu_B^2 < 0$ ,  $\mu_{k=0}^2/\mu_R^2 \approx 4.6$  for  $\Lambda = 7$ . Such an enhancement in the mass parameter originates from the systematic accumulation of higher loop contributions in the course of repeated blocking transformation which brings the theory to the IR regime. On the other hand, if  $\mu_B^2 > 0$  is chosen, the RG improved  $\mu_{k=0}^2$  will be comparable to the IMA result  $\mu_R^2$ .

The remarks on the running of the mass parameter also apply qualitatively to the flow pattern of the coupling strength which is displayed in Fig. 5. However, contrary to the former, there exists only a minute difference between the RG and IMA results. Therefore, the coupling constants obtained with various approaches near zero temperature are approximately equal. As we shall see in the next section, this is not the case for the large  $T$  limit, a regime where the IMA scheme becomes unreliable.

#### IV. HIGH TEMPERATURE LIMIT

We consider next the high temperature limit where  $T \gg \sqrt{k^2 + U''_{\beta,k}}$ . Since in this temperature range the second term in (2.9) dominates, if one neglects the first term in (2.9) entirely, the RG equation is simplified to

$$k \frac{\partial U_{\beta,k}(\Phi)}{\partial k} = -T \frac{k^3}{4\pi^2} \ln \left[ \frac{k^2 + U''_{\beta,k}(\Phi)}{k^2 + U''_{\beta,k}(0)} \right], \quad (4.1)$$

or

$$k \frac{\partial \bar{U}_k(\bar{\Phi})}{\partial k} = -\frac{k^3}{4\pi^2} \ln \left[ \frac{k^2 + \ddot{\bar{U}}_k(\bar{\Phi})}{k^2 + \ddot{\bar{U}}_k(0)} \right], \quad (4.2)$$

where  $\bar{U}_k(\bar{\Phi}) = \beta U_{\beta,k}(\Phi)$  and the dots denote differentiation with respect to  $\bar{\Phi} = \sqrt{\beta}\Phi$ , the new field variable defined in  $d = 3$ . One therefore concludes that the theory in this limit has undergone a dimensional reduction (DR) with temperature being completely decoupled,

leaving an effective three-dimensional theory described by (4.2). This is precisely the flow equation one would obtain for a three dimensional theory at  $T = 0$ . In other words, the high  $T$  behavior of the theory in  $d = 4$  corresponds to that of  $d = 3$  at zero temperature. Physically this phenomenon can be explained by noting that the high temperature limit is characterized by the shrinking of the “imaginary time” dimension having a period  $\beta$ . This in turn implies a suppression of the  $n \neq 0$  non-static modes in the Matsubara summation, thereby giving vanishing contribution to  $U_{\beta,k}(\Phi)$ . By neglecting the non-static modes completely, the remaining static sector is what the three-dimensional theory parameterized by  $\bar{U}_k(\bar{\Phi})$  describes. Notice that the coupling constant would become  $\bar{\lambda}_R = \lambda_R T$  carrying the dimension of mass.

The above analysis demonstrates that such a dimensional reduction takes place only if  $(k^2 + U''_{\beta,k}(\Phi))/T^2 \rightarrow 0$ , where  $U''_{\beta,k}(0)$  is the thermal mass parameter  $\mu_{\beta,k}^2$ . The condition also implies a small momentum scale  $k \ll T$ . However, as noted in [15], DR strictly does not takes place in the infinite temperature limit because  $\mu_{\beta,k}^2$  acquires a  $T^2$ -dependent correction in the leading order which renders the ratio  $U''_{\beta,k}(\Phi)/T^2$  finite for all  $T$ . In Fig. 7, we compare the parameters obtained from (4.2) with that of the full RG equation (2.9). In both mass and coupling constant, we see that the two results differ by a finite constant gap which persists to arbitrary large value of  $T$ . This indeeds supports the conclusion drawn in [15]. Nevertheless, for sufficiently large  $T$ , the theory exhibits a “partial” dimensional reduction since the minute difference can be neglected. We also notice that the value of  $\lambda_\beta$  at  $T = 0$  predicted by the DR prescription corresponds to the bare coupling constant  $\lambda_B > \tilde{\lambda}_R$ . One must remember, however, that the results obtained from the dimensionally reduced prescription for low  $T$  are not reliable.

Another interesting issue one may explore in the high temperature limit is to examine the behavior of the effective thermal coupling constant  $\lambda_{\beta,k}$  [16] and demonstrate the ineffectiveness of the one-loop IMA scheme. To see how perturbation theory breaks down, we notice first that from the IMA, the effective scale-dependent thermal parameters can be written as [11]:

$$\begin{aligned}\tilde{\mu}_{\beta,k}^2 &= \tilde{\mu}_k^2 + \frac{\tilde{\lambda}_R}{4\pi^2\beta^2} \int_{\beta\sqrt{k^2+\tilde{\mu}_R^2}}^{\infty} dx \frac{\sqrt{x^2 - \beta^2\tilde{\mu}_R^2}}{e^x - 1} \\ &= \tilde{\mu}_k^2 + \frac{\tilde{\lambda}_R}{4\pi^2\beta^2} \int_0^{\infty} dz \frac{z\sqrt{z^2 + b^2}}{\sqrt{z^2 + a^2}} \left(e^{\sqrt{z^2+a^2}} - 1\right)^{-1},\end{aligned}\tag{4.3}$$

and

$$\begin{aligned}\tilde{\lambda}_{\beta,k} &= \tilde{\lambda}_k - \frac{3\tilde{\lambda}_R^2}{8\pi^2} \int_{\beta\sqrt{k^2+\tilde{\mu}_R^2}}^{\infty} dx \frac{\sqrt{x^2 - \beta^2\tilde{\mu}_R^2}}{x^2} \frac{e^x - 1 + xe^x}{(e^x - 1)^2} \\ &= \tilde{\lambda}_k - \frac{3\tilde{\lambda}_R^2}{8\pi^2} \int_0^{\infty} dz \frac{z}{\sqrt{(z^2 + a^2)(z^2 + b^2)}} \left(e^{\sqrt{z^2+a^2}} - 1\right)^{-1},\end{aligned}\tag{4.4}$$

where  $a^2 = \beta^2(k^2 + \tilde{\mu}_R^2)$  and  $b^2 = \beta^2 k^2$  and the last integral expression in (4.4) is obtained via an integration by part. In the above,

$$\begin{aligned} \tilde{\mu}_k^2 = \tilde{\mu}_R^2 - \frac{\tilde{\lambda}_R}{64\pi^2\sqrt{k^2 + \tilde{\mu}_R^2}} & \left\{ 4k^3 + \tilde{\mu}_R^2 \left( 3k - \sqrt{k^2 + \tilde{\mu}_R^2} \right) - \frac{\tilde{\mu}_R^4}{k + \sqrt{k^2 + \tilde{\mu}_R^2}} \right. \\ & \left. - 4\mu_R^2\sqrt{k^2 + \mu_R^2} \ln\left( \frac{k + \sqrt{k^2 + \mu_R^2}}{\mu_R} \right) \right\} \end{aligned} \quad (4.5)$$

and

$$\tilde{\lambda}_k = \tilde{\lambda}_R - \frac{3\tilde{\lambda}_R^2}{16\pi^2} \left\{ \frac{k[2k + (2k^2 + \tilde{\mu}_R^2)(k^2 + \tilde{\mu}_R^2)^{-1/2}]}{(k + \sqrt{k^2 + \tilde{\mu}_R^2})^2} - \ln\left( \frac{k + \sqrt{k^2 + \tilde{\mu}_R^2}}{\tilde{\mu}_R} \right) \right\} \quad (4.6)$$

represent the running parameters at  $T = 0$ . The integrals appearing in (4.3) and (4.4) from finite temperature contribution can be approximated in the limits of large or small  $a$  and  $b$ . The details are provided in the Appendix following the classic treatment by Dolan and Jackiw [4]. With (A.15) and (A.34) in the Appendix, the one-loop approximation for the thermal parameters in the small  $a$  and  $b$  limits take on the forms

$$\begin{aligned} \tilde{\mu}_{\beta,k}^2 = \tilde{\mu}_k^2 + \frac{\tilde{\lambda}_R T^2}{24} + \frac{\tilde{\lambda}_R \tilde{\mu}_R^2}{16\pi^2} - \frac{3\tilde{\lambda}_R k^2}{16\pi^2} - \frac{\tilde{\lambda}_R \tilde{\mu}_R T}{4\pi^2} & \left[ \frac{\pi}{2} - \sin^{-1}\left( \frac{k}{\sqrt{k^2 + \tilde{\mu}_R^2}} \right) \right] \\ - \frac{\tilde{\lambda}_R \tilde{\mu}_R^2}{16\pi^2} & \left[ \ln\left( \frac{\sqrt{k^2 + \tilde{\mu}_R^2}}{4\pi T} \right) + \gamma + \tanh^{-1}\left( \frac{k}{\sqrt{k^2 + \tilde{\mu}_R^2}} \right) \right], \end{aligned} \quad (4.7)$$

and

$$\begin{aligned} \tilde{\lambda}_{\beta,k} = \tilde{\lambda}_k - \frac{3\tilde{\lambda}_R^2 T}{8\pi^2 \tilde{\mu}_R} & \left[ \frac{\pi}{2} - \sin^{-1}\left( \frac{k}{\sqrt{k^2 + \tilde{\mu}_R^2}} \right) \right] + \frac{3\tilde{\lambda}_R^2 k}{32\pi^2 T} \\ - \frac{3\tilde{\lambda}_R^2}{16\pi^2} & \left[ \ln\left( \frac{\sqrt{k^2 + \tilde{\mu}_R^2}}{4\pi T} \right) + \gamma + \tanh^{-1}\left( \frac{k}{\sqrt{k^2 + \tilde{\mu}_R^2}} \right) \right]. \end{aligned} \quad (4.8)$$

Eq. (4.7) shows that at sufficiently high  $T$  the mass parameter grows quadratically with  $T$ . However, the presence of  $-3\tilde{\lambda}_R k^2/16\pi^2$  tends to decrease  $\tilde{\mu}_{\beta,k}^2$ . Again, we observe a competition between the internal scale  $k$  and the external parameter  $T$ .

On the other hand, for large  $T$ , one naively expects the negative linear  $T$ -dependent term to dominate (4.8) giving rise to a vanishing or even negative  $\tilde{\lambda}_{\beta,k}$ . However, it is a well known fact that the correct high  $T$  behavior of  $\tilde{\lambda}_{\beta,k}$  cannot be accounted for by the simple one-loop result (4.8), and that when higher loop contributions such as daisy and superdaisy graphs are incorporated, the linear term will be suppressed. Since it is a nontrivial task to resum these higher loop effects, various attempts based on the use of gap equations [17], the Schwinger-Dyson equation [18] or RG have been made to provide an effective resummation. However, the details of resummation have raised some concerns [19]. We shall now illustrate how the RG equation obtained in (2.9) can be utilized to address these issues.

As explained in sec. II, a resummation of all possible non-overlapping diagrams can be achieved with our RG equation (2.9). By comparing (2.9) with (2.8), we see that the difference between the IMA and the RG prescriptions is due to the fact that in the latter, instead of using the  $k$ -independent  $V(\Phi)$ ,  $U_{\beta,k}(\Phi)$  is employed to give the scale-dependent thermal parameters  $\tilde{\mu}_{\beta,k}^2$  and  $\tilde{\lambda}_{\beta,k}$  which are then used in the evaluation of  $U_{\beta,k-\Delta k}(\Phi)$ . By iterating this procedure, higher loop contributions are automatically taken into account. Guided by this logic, we proceed to improve (4.7) and (4.8) by replacing the right-hand-side of the expressions by the effective  $\tilde{\mu}_{\beta,k}^2$  and  $\tilde{\lambda}_{\beta,k}$ . This leads to the following set of two coupled equations in the limit of small  $a = \beta(k^2 + \mu_{\beta,k}^2)^{1/2}$ :

$$\begin{aligned} \mu_{\beta,k}^2 = & \mu_k^2 + \frac{\lambda_{\beta,k} T^2}{24} + \frac{\lambda_{\beta,k} \mu_{\beta,k}^2}{16\pi^2} - \frac{\lambda_{\beta,k} \mu_{\beta,k} T}{4\pi^2} \left[ \frac{\pi}{2} - \sin^{-1} \left( \frac{k}{\sqrt{k^2 + \mu_{\beta,k}^2}} \right) \right] \\ & - \frac{3\lambda_{\beta,k} k^2}{16\pi^2} - \frac{\lambda_{\beta,k} \mu_{\beta,k}^2}{16\pi^2} \left[ \ln \left( \frac{\sqrt{k^2 + \mu_{\beta,k}^2}}{4\pi T} \right) + \gamma + \tanh^{-1} \left( \frac{k}{\sqrt{k^2 + \mu_{\beta,k}^2}} \right) \right], \end{aligned} \quad (4.9)$$

and

$$\begin{aligned} \lambda_{\beta,k} = & \lambda_k - \frac{3\lambda_{\beta,k}^2 T}{8\pi^2 \mu_{\beta,k}} \left[ \frac{\pi}{2} - \sin^{-1} \left( \frac{k}{\sqrt{k^2 + \mu_{\beta,k}^2}} \right) \right] + \frac{3\lambda_{\beta,k}^2 k}{32\pi^2 T} \\ & - \frac{3\lambda_{\beta,k}^2}{16\pi^2} \left[ \ln \left( \frac{\sqrt{k^2 + \mu_{\beta,k}^2}}{4\pi T} \right) + \gamma + \tanh^{-1} \left( \frac{k}{\sqrt{k^2 + \mu_{\beta,k}^2}} \right) \right]. \end{aligned} \quad (4.10)$$

These coupled equations are actually inferred by (2.10) and (2.11). The only difference is that we have truncated the higher order contributions in (4.10) and (4.9). Notice that they are slightly different from those described by Chia [17] in that the continuous feedbacks from  $\mu_{\beta,k}^2$  to  $\lambda_{\beta,k}$  and vice versa are systematically incorporated, hence leading to a more accurate determination of the high temperature behavior of the theory.

Taking the  $k = 0$  limit for simplicity, we have

$$\mu_\beta^2 = \mu_R^2 + \frac{\lambda_\beta T^2}{24} - \frac{\lambda_\beta \mu_\beta T}{8\pi} - \frac{\lambda_\beta \mu_\beta^2}{16\pi^2} \left[ \ln \left( \frac{\mu_\beta}{4\pi T} \right) + \gamma - 1 \right], \quad (4.11)$$

and

$$\lambda_\beta = \lambda_R - \frac{3\lambda_\beta^2 T}{16\pi \mu_\beta} - \frac{3\lambda_\beta^2}{16\pi^2} \left[ \ln \left( \frac{\mu_\beta}{4\pi T} \right) + \gamma \right], \quad (4.12)$$

where the subscript  $k$  is dropped for brevity. From (4.11), we notice the quadratic  $T$  dependence of the thermal mass parameter  $\mu_\beta^2$ . To probe the high  $T$  behavior of  $\lambda_\beta$ , however, would require a more precise determination of the ratio  $\mathcal{R} = \mu_\beta/T$  present in the equation for  $\lambda_\beta$ . In the one-loop approximation, the use of  $\mu_\beta = \mu_R$  for (4.12) leads to a rapid linear decrease of  $\lambda_\beta$  with  $T$ , signalling the breakdown of perturbation theory. By

including the dominant  $\lambda_\beta T^2/24$  term which gives  $\mathcal{R} \rightarrow \lambda_\beta^{1/2}/2\sqrt{6}$  as  $T \rightarrow \infty$ , we obtain from (4.12) the relation

$$\lambda_R = \lambda_\beta + \frac{3\sqrt{6}}{8\pi} \lambda_\beta^{3/2} + \dots, \quad (4.13)$$

which for  $\lambda_R = 0.1$  takes on the value  $\lambda_\beta = 0.09186$ . Further refinement with  $\mathcal{R} = \lambda_\beta^{1/2}/2\sqrt{6} - 3\lambda_\beta/48\pi$  yields

$$\lambda_R = \lambda_\beta + \frac{3\lambda_\beta^2}{16\pi} \left( \frac{\lambda_\beta^{1/2}}{2\sqrt{6}} - \frac{3\lambda_\beta}{48\pi} \right)^{-1/2} + \dots, \quad (4.14)$$

or  $\lambda_\beta = 0.09160$ . This is in accord with that illustrated in Fig. 8 for the  $T$  dependence of the thermal parameters  $\mu_\beta$  and  $\lambda_\beta$ . Thus, we conclude that at very high  $T$ ,  $\lambda_\beta$  approaches a constant nonvanishing positive value. In Fig. 7, the infinitesimal decrease of  $\lambda_\beta$  in the high  $T$  regime is due to the nature of the coupled equations in which a particular value of  $\lambda_\beta$  is first used in (4.11) for deducing  $\mathcal{R}$  which is subsequently inserted to (4.12) for the determination of a new improved  $\lambda_\beta$ . In general, the more accurately  $\mathcal{R}$  is, the better the agreement with that generated by (2.9). We emphasize here that contrary to the claim in [19],  $\lambda_\beta$  can never increase with  $T$  because of the negative sign associated with the temperature-dependent term in the full RG flow equation (2.9). The inclusion of higher loop effects can only modify the leading  $T$ -dependent behavior, but not the sign. When RG is invoked to study  $\lambda_{\beta,k}$ , it is crucial to keep in mind that there are three dimensionless combinations  $b = k/T$ ,  $b' = \tilde{\mu}_R/k$  and  $b'' = \tilde{\mu}_R/T$  present in the flow equation with  $b'$  and  $b''$  having opposite effect to the running of  $\lambda_{\beta,k}$ . Erroneous use of  $b''$  in the RG analysis can lead to the wrong claim that  $\lambda_{\beta,k}$  rises with  $T$ . By employing the correct choice  $b'$ , one will reproduce the standard result that  $\lambda_{\beta,k}$  increases logarithmically with  $k$ .

We also notice that the mass parameter  $\mu_\beta^2$  obtained with the RG method is slightly smaller than that of the one-loop result. The reason again is due to the continuous feedback of the higher loop effects as well as the use of improved value for the thermal coupling  $\lambda_\beta$  which is smaller compared to  $\lambda_R$ .

Since the scales  $k$  and  $T$  enter in a complementary manner, in the regime where  $k \gg T$  one expects the thermal effects to be suppressed. This can be seen by noting that

$$\tilde{\mu}_{\beta,k}^2 = \tilde{\mu}_k^2 + \frac{\tilde{\lambda}_R}{4\pi^2\beta^2} e^{-a} \left[ 1 + \frac{a}{2} + \frac{b^2}{2a} + \dots \right], \quad (4.15)$$

and

$$\tilde{\lambda}_{\beta,k} = \tilde{\lambda}_k - \frac{3\tilde{\lambda}_R^2}{8\pi^2} \frac{e^{-a}}{a} \left[ 1 + a - \frac{1}{2}(a^2 - b^2) + \dots \right] \quad (4.16)$$

with the help of (A.19) and (A.37). Indeed, the thermal parameters are suppressed exponentially by a factor  $e^{-a}$ . When the blocking scale  $k$  coincides with the UV cut-off  $\Lambda$ , all thermal contributions must vanish since for  $k = \Lambda$  the original temperature-independent bare theory is recovered. Equivalently, one may say that the counterterms are independent of temperature. This implies that we must have  $e^{-\beta\Lambda} \rightarrow 0$  in principle. Correspondingly, we work within a temperature range which lies sufficiently far from  $\Lambda$  to ensure that nearly

all particles will have a momentum below  $\Lambda$  according to the Bose-Einstein distribution. The suppression of thermal effects for large  $\beta k$  can indeed be seen from Fig. 4.

## V. SPONTANEOUS BREAKING AND RESTORATION OF SYMMETRY

Turning to the scenario of spontaneous symmetry breaking (SSB) with  $\mu_R^2 < 0$ , the one-loop finite temperature blocked potential becomes

$$\begin{aligned} \tilde{U}_{\beta,k}(\Phi) = & \frac{\tilde{\mu}_R^2}{2} \Phi^2 \left(1 - \frac{\tilde{\lambda}_R}{64\pi^2}\right) + \frac{\tilde{\lambda}_R}{4!} \Phi^4 \left[1 - \frac{\tilde{\lambda}_R}{256\pi^2} \frac{36\tilde{\mu}_R^4 + 84\tilde{\lambda}_R\tilde{\mu}_R^2 M^2 + 25\tilde{\lambda}_R^2 M^4}{(\tilde{\mu}_R^2 + \tilde{\lambda}_R M^2/2)^2}\right] \\ & + \frac{1}{64\pi^2} \left\{ -2k \left(2k^2 + \tilde{\mu}_R^2 + \frac{1}{2}\tilde{\lambda}_R \Phi^2\right) \left(k^2 + \tilde{\mu}_R^2 + \frac{1}{2}\tilde{\lambda}_R \Phi^2\right)^{1/2} \right. \\ & + \left. \left(\tilde{\mu}_R^2 + \frac{1}{2}\tilde{\lambda}_R \Phi^2\right)^2 \ln \left[ \frac{2k^2 + \tilde{\mu}_R^2 + \tilde{\lambda}_R \Phi^2/2 + 2k\sqrt{k^2 + \tilde{\mu}_R^2 + \tilde{\lambda}_R \Phi^2/2}}{\tilde{\mu}_R^2 + \tilde{\lambda}_R M^2/2} \right] \right\} \\ & + \frac{1}{2\pi^2\beta} \int_k^\Lambda dp p^2 \ln \left[1 - e^{-\beta\sqrt{p^2 + \tilde{\mu}_R^2 + \tilde{\lambda}_R \Phi^2/2}}\right], \end{aligned} \quad (5.1)$$

where we have chosen the following off-shell renormalization conditions used in [13]:

$$\begin{cases} \tilde{\mu}_R^2 = \text{Re} \left[ \frac{\partial^2 \tilde{U}_{\beta,k}}{\partial \Phi^2} \Big|_{\Phi=\beta^{-1}=k=0} \right] < 0 \\ \tilde{\lambda}_R = \frac{\partial^4 \tilde{U}_{\beta,k}}{\partial \Phi^4} \Big|_{\Phi=M, \beta^{-1}=k=0}, \end{cases} \quad (5.2)$$

where  $M = \langle \Phi \rangle$  is the nonvanishing vacuum expectation at which  $\tilde{U}_{\beta,k}(\Phi)$  is minimized. Since  $\tilde{\mu}_R^2 < 0$ , it can no longer be interpreted as the mass parameter for the theory. Instead, the mass parameter is determined by  $\tilde{U}_{\beta^{-1}=k=0}''(\Phi = M) \approx -2\tilde{\mu}_R^2 > 0$ . A characteristic feature of SSB is the development of an imaginary sector in  $\tilde{U}_{\beta,k}(\Phi)$  in the regime where  $k$  and  $\Phi$  are small and the argument  $k^2 + \tilde{\mu}_R^2 + \tilde{\lambda}_R \Phi^2/2$  inside the logarithm and the square root becomes negative. Any attempt to extract from the low temperature range the critical temperature  $T_c$  beyond which symmetry restoration takes place using (5.1) is fruitless since the resulting  $T_c$  is complex due to the presence of an imaginary part in the finite temperature contribution found in the last line of (5.1) [4]. However, concentrating only on the real part of  $U_{\beta,k}(\Phi)$  and taking into account the higher order daisy and super-daisy graphs, Dolan and Jackiw [4] obtained a critical temperature  $\tilde{T}_c$

$$\tilde{T}_c = \left( \frac{-24\tilde{\mu}_R^2}{\tilde{\lambda}_R} \right)^{1/2} \quad (5.3)$$

beyond which symmetry restoration takes place via a second order phase transition. Nevertheless, a *true* symmetry restoration should be accompanied by the disappearance and not

negligence of the imaginary contribution in the thermal blocked potential  $U_{\beta,k}(\Phi)$ . This is clearly not possible within the context of the IMA since the finite temperature effects never enter the argument  $k^2 + \tilde{\mu}_R^2 + \tilde{\lambda}_R \Phi^2/2$ , and we see that the imaginary contribution persists for all  $T$  unless higher loop effects daisy and super-daisy are properly taken into account [4].

It is interesting to compare our finite temperature RG prescription with that used in [4] since in our approach the argument inside the logarithm and square root becomes  $k^2 + U''_{\beta,k}(\Phi)$ , and is a positive quantity beyond  $T_c$ .

With the RG flow equation (2.9), we define  $T_c$  to be the symmetry restoration temperature above which the imaginary contribution vanishes entirely. A comparison between  $T_c$  and  $\tilde{T}_c$  is illustrated in Fig. 9. Notice that in the RG scheme,  $\mu_R^2$  is used instead of  $\tilde{\mu}_R^2$  which together with  $\tilde{\lambda}_R$  determines  $\tilde{T}_c$ . For sufficiently low  $-\mu_R^2$ , one finds agreement between the RG result and (5.3). However, they begin to deviate for large  $-\mu_R^2$  and the RG approach gives a higher symmetry restoration temperature compared to  $\tilde{T}_c$  in this limit. This observation can be explained by noting that for a given  $T$ , the RG prescription yields a lower value for the thermal mass parameter  $\mu_\beta^2$  compared to that of [4] for large  $T$  due to the fact that we also take into consideration the evolution of the coupling constant  $\lambda_\beta$ . Therefore, for the condition  $\mu_{\beta_c}^2 = 0$  to be satisfied, we must have  $T_c \geq \tilde{T}_c$ . Had we replaced  $\lambda_\beta$  by  $\lambda_R$  by neglecting the influence of the running of the coupling constant on the mass gap equation (4.11), we would have  $T_c = \tilde{T}_c$  instead.

Notice that it is also possible to extract  $T_c$  without confronting the complication of imaginary contribution by starting from a temperature value  $T > T_c$  and gradually lower the temperature. Both methods lead to the same  $T_c$ , as has been checked numerically. Near the critical point  $T \gtrsim T_c$ , one obtains from (4.11):

$$\mu_\beta \approx \frac{\pi T}{3} \left[ 1 - \frac{1}{\lambda_\beta T^2} \lim_{T \rightarrow T_c} (\lambda_\beta T^2) \right], \quad (5.4)$$

such that  $\mu_{\beta_c} = 0$  at exactly  $T = T_c$ . Similarly, a resummation of higher physical loop contributions in (4.12) shows that in the vicinity of critical point, the coupling constant behaves as:

$$\lambda_\beta \approx \frac{\lambda_R}{1 + \frac{3\lambda_R T}{16\pi \mu_\beta}} \longrightarrow \frac{16\pi^2}{9} \left[ 1 - \frac{1}{\lambda_\beta T^2} \lim_{T \rightarrow T_c} (\lambda_\beta T^2) \right], \quad (5.5)$$

after substituting (5.4). One therefore sees that the effective thermal coupling constant  $\lambda_\beta$  vanishes at the transition temperature and the theory becomes non-interacting. This can also be obtained directly from (4.4) which implies

$$\frac{3\mathcal{I}}{8\pi^2} \lambda_{\beta_c}^2 + \lambda_{\beta_c} - \lambda_R = 0 \quad (5.6)$$

at precisely  $T = T_c$ , where

$$\mathcal{I} = \int_0^\infty \frac{dz}{z(e^z - 1)}. \quad (5.7)$$

That  $\mathcal{I}$  diverges naturally gives  $\lambda_{\beta_c} = 0$ . The temperature dependences of  $\langle \Phi \rangle$ ,  $\mu_\beta^2$  and  $\lambda_\beta$  above and below  $T_c$  are depicted in Fig. 10. Our numerical result indeed yields  $\lambda_{\beta_c} = 0$

when approaching  $T_c$  both from above and below. However, for  $T \lesssim T_c$  in the SSB phase, there exists certain numerical fluctuations which nevertheless diminish at  $T_c$ . A second order phase transition characterized by a continuous decrease of the vacuum condensate is predicted with our RG flow equation. With  $\lambda_{\beta_c} = 0$  at the transition point, the IR divergence is completely lifted. Had we not incorporated the higher loop effects, the IR singularity would persist and result in a breakdown of the perturbation theory.

The inapplicability of the IMA scheme near  $T_c$  can be illustrated by studying the order of transition it predicts. For  $T \gtrsim T_c$ , with  $\mu_\beta^2 \gtrsim 0$  and  $\lambda_\beta < 0$ , a first order transition is obtained due to a positive higher order  $\Phi^6$  contribution, in contradiction with that predicted with RG. Our analyses are in agreements with that obtained in [10] and [17]. However, in [10] where a smooth momentum regulator is used, there exists residual dependence of the running parameters on the shape of the momentum regularizing function.

Physically the critical temperature corresponds to a fixed point in the RG trajectory. Since this fixed point is of Gaussian nature with  $\mu_{\beta_c}^2 = \lambda_{\beta_c} = 0$ , any interaction between the scalar fields must be of higher order and parameterized by the irrelevant operators classified around the fixed point. For this theory, the critical exponents can be accurately determined and shown to coincide with that of the three-dimensional theory at  $T = 0$  [10]. From Fig. 10, we also observe that after symmetry restoration with  $T > T_c$ ,  $\lambda_\beta$  rises again and eventually approaches the same constant value as that without going through phase transition.

## VI. SUMMARY AND DISCUSSIONS

In this paper we have carried out the investigation of finite temperature scalar theory using an improved RG program. Our RG flow equation has successfully reproduced the characteristic behaviors of the system in both high and low  $T$  limits. In particular, we have established a connection between the  $O(4)$  and  $O(3)$  blocked potentials in large  $k$  and small  $k$  regimes in sec III, and in sec IV dimensional reduction and the proper high temperature RG flow patterns for  $\mu_{\beta,k}^2$  and  $\lambda_{\beta,k}$  are deduced. At  $k = 0$  the coupling constant  $\lambda_{\beta,k}$  is seen to decrease with  $T$  and approximately approaches a positive constant value. Thermal effects are suppressed for large  $k$  and vanish entirely at  $k = \Lambda$  where the bare theory is defined. In addition we demonstrated in Sec. V how true symmetry restoration is attained with vanishing imaginary contribution in  $U_{\beta,k}(\Phi)$ .

Our method is more advantageous compared to the approaches mentioned in the Introduction in a number of ways: The coupled equations (4.9) and (4.10) derived from (2.9) for  $\mu_{\beta,k}^2$  and  $\lambda_{\beta,k}$  have provided further improvement compared to that derived for  $\mu_\beta^2$  alone in [4] and [17] since the continuous feedbacks between  $\mu_{\beta,k}^2$  and  $\lambda_{\beta,k}$  are retained systematically. In addition, through (2.9), we are able to analyze simultaneously the competing effects of two scales, namely,  $k$  and  $T$  in an unambiguous manner. A RG approach based on the running of an internal scale  $k$  is more “physical” than that employed in [6] with the external  $T$  as the running parameter. When applied to the Yang-Mills theory with a momentum-dependent gluon polarization tensor, the resummation using the latter scheme is inconsistent [6]. This poses no difficulty with RG formulation which naturally includes both the temperature and momentum effects.

Various interesting issues can be explored with our RG scheme in light of its success. For example, one can use this approach to study the nature of the phase transition for the electroweak theory. For  $\lambda_R \ll g_R^2$ , where  $g_R$  is the coupling constant for the gauge fields, one would expect a first order transition which is required for explaining the asymmetry of the baryogenesis [20]. For the Yang-Mills theory, a RG flow equation similar to (2.9) will provide information on the roles of  $k$  and  $T$  on the running of the gauge coupling constant. The flow of the theory with at  $T = 0$  has been worked out [21]. It would be interesting to investigate the effect of  $T$  on such a theory which is known to exhibit asymptotic freedom at  $T = 0$ . If  $T$  and  $k$  can generate opposite effects as for the scalar theory, there will be nontrivial consequences on the picture of deconfinement transition of quarks and gluons. In addition, the resummation of “hot thermal” loops using this RG approach will readily yield the gauge-independent gluon damping rate and be compared with that obtained in [5] via an effective action. Works along these directions are currently in progress.

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## APPENDIX

We proceed to evaluate the integrals shown in (4.3) and (4.4) following the techniques utilized in [4]. The presence of the blocking scale  $k$  will slightly complicate the algebra compared with the previous studies. We first consider the small  $a$  limit of the integral in (4.4) having the form

$$\begin{aligned}
I(a, b) &= \int_0^\infty dz \frac{z}{\sqrt{(z^2 + a^2)(z^2 + b^2)}} \left( e^{\sqrt{z^2 + a^2}} - 1 \right)^{-1} \\
&= \frac{1}{2} \int_0^\infty dz \frac{z}{\sqrt{(z^2 + a^2)(z^2 + b^2)}} \left[ \coth \left( \frac{\sqrt{z^2 + a^2}}{2} \right) - 1 \right] \\
&= \frac{1}{2} \int_0^\infty dz \frac{z}{\sqrt{(z^2 + a^2)(z^2 + b^2)}} \left[ 2\sqrt{z^2 + a^2} \sum_{n=-\infty}^\infty \frac{1}{z^2 + a^2 + 4\pi^2 n^2} - 1 \right] \\
&= I^{(1)}(a, b) + I^{(2)}(a, b) ,
\end{aligned} \tag{A.1}$$

where

$$I^{(1)}(a, b) = \int_0^\infty dz \frac{z}{\sqrt{(z^2 + b^2)}} \sum_{n=-\infty}^\infty \frac{1}{z^2 + a^2 + 4\pi^2 n^2}, \tag{A.2}$$

and

$$I^{(2)}(a, b) = -\frac{1}{2} \int_0^\infty dz \frac{z}{\sqrt{(z^2 + a^2)(z^2 + b^2)}}. \quad (\text{A.3})$$

Since divergences are encountered when splitting the integral in such a manner, we introduce a suppression factor  $z^{-\epsilon}$  to regularize the individual sum and expect the infinities to cancel, thereby making the final result for (A.1) finite. By rewriting (A.2) as

$$I_\epsilon^{(1)}(a, b) = \sum_{n=-\infty}^{\infty} \left( a^2 + 4\pi^2 n^2 \right)^{-\epsilon/2} \int_0^\infty dx \frac{x^\epsilon}{(1+x^2)\sqrt{b^2 x^2 + a^2 + 4\pi^2 n^2}}, \quad (\text{A.4})$$

via a change of variable  $x = (a^2 + 4\pi^2 n^2)^{1/2} z^{-1}$ , the integration can be carried out with the help of

$$\begin{aligned} \int_0^\infty dx \frac{x^\epsilon}{(1+x^2)\sqrt{b^2 x^2 + c^2}} &= \frac{b^{1-\epsilon} c^{-2+\epsilon}}{2\sqrt{\pi}} \Gamma\left(1 - \frac{\epsilon}{2}\right) \Gamma\left(\frac{-1+\epsilon}{2}\right) F\left(1, 1 - \frac{\epsilon}{2}, \frac{3-\epsilon}{2}, \frac{b^2}{c^2}\right) \\ &+ \frac{\pi \sec(\epsilon\pi/2)}{2\sqrt{c^2 - b^2}}, \end{aligned} \quad (\text{A.5})$$

where

$$F(a, b, c; \gamma) = F(b, a, c; \gamma) = B^{-1}(b, c-b) \int_0^1 dx x^{b-1} (1-x)^{c-b-1} (1-\gamma x)^{-a}, \quad (\text{A.6})$$

with

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 dt t^{x-1} (1-t)^{y-1}. \quad (\text{A.7})$$

Substitution of (A.5) into (A.4) then gives

$$\begin{aligned} I_\epsilon^{(1)}(a, b) &= \frac{\pi}{2} \sec\left(\frac{\epsilon\pi}{2}\right) \left\{ \frac{a^{-\epsilon}}{\sqrt{a^2 - b^2}} + 2 \sum_{n=1}^{\infty} \frac{(a^2 + 4\pi^2 n^2)^{-\epsilon/2}}{\sqrt{a^2 + 4\pi^2 n^2 - b^2}} \right\} \\ &+ \frac{b^{1-\epsilon}}{2\sqrt{\pi}} \Gamma\left(1 - \frac{\epsilon}{2}\right) \Gamma\left(\frac{-1+\epsilon}{2}\right) \left\{ a^{-2} F\left(1, 1 - \frac{\epsilon}{2}, \frac{3-\epsilon}{2}, \frac{b^2}{a^2}\right) \right. \\ &\left. + 2 \sum_{n=1}^{\infty} (a^2 + 4\pi^2 n^2)^{-1} F\left(1, 1 - \frac{\epsilon}{2}, \frac{3-\epsilon}{2}, \frac{b^2}{a^2 + 4\pi^2 n^2}\right) \right\}, \end{aligned} \quad (\text{A.8})$$

which in the limit of vanishing  $\epsilon$  becomes

$$\begin{aligned} I_\epsilon^{(1)}(a, b) &= \frac{\pi}{2} \frac{1}{\sqrt{a^2 - b^2}} + 2^{-1-\epsilon} \pi^{-\epsilon} \zeta(1+\epsilon) + \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \left[ \left(1 + \frac{a^2 - b^2}{4\pi^2 n^2}\right)^{-1/2} - 1 \right] \\ &- \frac{1}{\sqrt{a^2 - b^2}} \sin^{-1}\left(\frac{b}{a}\right) - 2 \sum_{n=1}^{\infty} \left(a^2 + 4\pi^2 n^2 - b^2\right)^{-1/2} \sin^{-1}\left(\frac{b}{\sqrt{a^2 + 4\pi^2 n^2}}\right) \\ &= \frac{1}{2\epsilon} + \frac{\pi}{2} \frac{1}{\sqrt{a^2 - b^2}} + \frac{1}{2} (\gamma - \ln 2\pi) - \frac{1}{\sqrt{a^2 - b^2}} \sin^{-1}\left(\frac{b}{a}\right) - \frac{b}{12} + O(a^2), \end{aligned} \quad (\text{A.9})$$

where we have used

$$\zeta(1 + \epsilon) = \sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon}} = -\frac{2^\epsilon \pi^{1+\epsilon} \zeta(-\epsilon)}{\Gamma(1 + \epsilon) \sin(\frac{\epsilon\pi}{2})} = (2\pi)^\epsilon \left[ \frac{1}{\epsilon} - \ln 2\pi + \gamma + O(\epsilon) \right], \quad (\text{A.10})$$

$$F\left(1, 1, \frac{3}{2}, x^2\right) = \frac{\sin^{-1} x}{x\sqrt{1-x^2}}, \quad (\text{A.11})$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \left( a^2 + 4\pi^2 n^2 - b^2 \right)^{-1/2} \sin^{-1} \left( \frac{b}{\sqrt{a^2 + 4\pi^2 n^2}} \right) \\ & \approx b \sum_{n=1}^{\infty} \left( a^2 + 4\pi^2 n^2 \right)^{-1} \left[ 1 - \frac{b^2}{a^2 + 4\pi^2 n^2} \right]^{-1/2} = b \sum_{n=1}^{\infty} \left( a^2 + 4\pi^2 n^2 \right)^{-1} + \dots \\ & = b \left[ \frac{1}{4a} \coth \left( \frac{a}{2} \right) - \frac{1}{2a^2} \right] = \frac{b}{24} + \frac{a^2 b}{1440} + \dots, \end{aligned} \quad (\text{A.12})$$

and neglected

$$\sum_{n=1}^{\infty} \frac{1}{n} \left[ \left( 1 + \frac{a^2 - b^2}{4\pi^2 n^2} \right)^{-1/2} - 1 \right] = -\frac{(a^2 - b^2)}{8\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^3} + \dots = O(a^2). \quad (\text{A.13})$$

In a similar manner, we have

$$\begin{aligned} I_\epsilon^{(2)}(a, b) &= -\frac{1}{2} \int_0^\infty dz \frac{z^{1-\epsilon}}{\sqrt{(z^2 + a^2)(z^2 + b^2)}} \\ &= -\frac{1}{4\sqrt{\pi}} \left\{ a^{-1} b^{1-\epsilon} \Gamma(1 - \frac{\epsilon}{2}) \Gamma(\frac{-1+\epsilon}{2}) F\left(\frac{1}{2}, 1 - \frac{\epsilon}{2}, \frac{3-\epsilon}{2}, \frac{b^2}{a^2}\right) \right. \\ &\quad \left. + a^{-\epsilon} \Gamma(\frac{1-\epsilon}{2}) \Gamma(\frac{\epsilon}{2}) F\left(\frac{1}{2}, \frac{\epsilon}{2}, \frac{1+\epsilon}{2}, \frac{b^2}{a^2}\right) \right\} \\ &\xrightarrow{\epsilon \rightarrow 0} \frac{1}{2} \tanh^{-1} \left( \frac{b}{a} \right) - \frac{1}{2\epsilon} + \frac{1}{2} \ln \frac{a}{2} + O(\epsilon). \end{aligned} \quad (\text{A.14})$$

Combining (A.9) and (A.14) yields the finite result:

$$\begin{aligned} I(a, b) &= \int_0^\infty dz \frac{z}{\sqrt{(z^2 + a^2)(z^2 + b^2)}} \left( e^{\sqrt{z^2 + a^2}} - 1 \right)^{-1} \\ &= \frac{1}{\sqrt{a^2 - b^2}} \left[ \frac{\pi}{2} - \sin^{-1} \left( \frac{b}{a} \right) \right] + \frac{1}{2} \left[ \ln \frac{a}{4\pi} + \gamma + \tanh^{-1} \left( \frac{b}{a} \right) \right] - \frac{b}{12} + O(a^2), \end{aligned} \quad (\text{A.15})$$

which for vanishing  $b$ , reduces to that obtained in [4].

On the other hand, for  $k \gg T \gg \mu_R$ , the original integral expression in (4.4) can be approximated as:

$$\begin{aligned}\tilde{I}(a, b) &= \int_a^\infty dx \frac{\sqrt{x^2 - (a^2 - b^2)}}{x^2} \frac{e^x - 1 + xe^x}{(e^x - 1)^2} \\ &= \int_a^\infty \frac{dx}{x} e^{-x} (1 + x) \left[ 1 - \frac{(a^2 - b^2)}{2x^2} + \dots \right] \\ &= e^{-a} \left[ 1 - \frac{1}{4}(1 + a) \frac{(a^2 - b^2)}{a^2} \right] - Ei(-a) \left[ 1 + \frac{(a^2 - b^2)}{4} \right] + \dots,\end{aligned}\tag{A.16}$$

where we have used

$$\int_a^\infty dx \frac{e^{-x}}{x^{n+1}} = \frac{(-1)^{n+1}}{n!} Ei(-a) + \frac{e^{-a}}{a^n} \sum_{m=0}^{n-1} \frac{(-1)^m a^m}{n(n-1) \dots (n-m)},\tag{A.17}$$

with

$$Ei(-a) = - \int_a^\infty \frac{dx}{x} e^{-x}.\tag{A.18}$$

Since  $Ei(-a) \rightarrow -e^{-a}/a$  as  $a \rightarrow \infty$ , we arrive at

$$\tilde{I}(a \rightarrow \infty, b) = \frac{e^{-a}}{a} \left[ 1 + a - \frac{1}{2}(a^2 - b^2) + \dots \right].\tag{A.19}$$

The other integral shown in (4.3) has the following form:

$$J(a, b) = \int_0^\infty dz \frac{z\sqrt{z^2 + b^2}}{\sqrt{z^2 + a^2}} \left( e^{\sqrt{z^2 + a^2}} - 1 \right)^{-1}.\tag{A.20}$$

Before evaluating (A.20) fully, we first take the limit  $b = 0$  and write

$$J(a) = J(a, 0) = \int_0^\infty dz \frac{z^2}{\sqrt{z^2 + a^2}} \left( e^{\sqrt{z^2 + a^2}} - 1 \right)^{-1}.\tag{A.21}$$

Following the procedures outlined above, eq. (A.21) becomes

$$\begin{aligned}J(a) &= \frac{\pi^2}{6} + \sum_{n=-\infty}^\infty \int dz \, z^2 \left[ \frac{1}{z^2 + a^2 + 4\pi^2 n^2} - \frac{1}{z^2 + 4\pi^2 n^2} \right] - \frac{1}{2} \int_0^\infty dz \left[ \frac{z^2}{\sqrt{z^2 + a^2}} - z \right] \\ &= \frac{\pi^2}{6} + J^{(1)}(a) + J^{(2)}(a),\end{aligned}\tag{A.22}$$

where we have added and subtracted

$$\begin{aligned}J(0) &= \int_0^\infty dz \frac{z}{e^z - 1} = \sum_{n=-\infty}^\infty \int dz \frac{z^2}{z^2 + 4\pi^2 n^2} - \frac{1}{2} \int_0^\infty dz \, z \\ &= \frac{\pi^2}{6}.\end{aligned}\tag{A.23}$$

Using

$$\int_0^\infty dz \frac{z^{2-\epsilon}}{(z^2 + \alpha_1^2)(z^2 + \alpha_2^2)} = \frac{\pi}{2(\alpha_1^2 - \alpha_2^2)} \sec\left(\frac{\epsilon\pi}{2}\right) (\alpha_1^{1-\epsilon} - \alpha_2^{1-\epsilon}), \quad (\text{A.24})$$

the first integral in (A.22) in its regularized form can be written as

$$\begin{aligned} J_\epsilon^{(1)}(a) &= -a^2 \sum_{n=-\infty}^{\infty} \int_0^\infty dz \frac{z^{2-\epsilon}}{(z^2 + 4\pi^2 n^2)(z^2 + 4\pi^2 n^2 + a^2)} \\ &= -\frac{\pi}{2} \sec\left(\frac{\epsilon\pi}{2}\right) \sum_{n=-\infty}^{\infty} \left[ (4\pi^2 n^2 + a^2)^{(1-\epsilon)/2} - (2\pi n)^{1-\epsilon} \right] \\ &= -\frac{\pi}{2} \sec\left(\frac{\epsilon\pi}{2}\right) \left\{ a^{1-\epsilon} + 2 \tilde{J}_\epsilon^{(1)}(a) \right\}, \end{aligned} \quad (\text{A.25})$$

where

$$\tilde{J}_\epsilon^{(1)}(a) = \sum_{n=1}^{\infty} \left[ (4\pi^2 n^2 + a^2)^{(1-\epsilon)/2} - (2\pi n)^{1-\epsilon} \right]. \quad (\text{A.26})$$

Since

$$\begin{aligned} \frac{\partial \tilde{J}_\epsilon^{(1)}(a)}{\partial a} &= a(1-\epsilon) \sum_{n=1}^{\infty} (4\pi^2 n^2 + a^2)^{-(1+\epsilon)/2} \\ &= a(1-\epsilon) \left\{ \frac{\zeta(1+\epsilon)}{(2\pi)^{1+\epsilon}} + \sum_{n=1}^{\infty} \frac{1}{(2\pi n)^{1+\epsilon}} \left[ \left(1 + \frac{a^2}{4\pi^2 n^2}\right)^{-(1+\epsilon)/2} - 1 \right] \right\} \\ &= a(1-\epsilon)(2\pi)^{-(1+\epsilon)} \zeta(1+\epsilon) + O(a^3), \end{aligned} \quad (\text{A.27})$$

this implies

$$\tilde{J}_\epsilon^{(1)}(a) = \frac{a^2}{2} (1-\epsilon)(2\pi)^{-(1+\epsilon)} \zeta(1+\epsilon) + \dots, \quad (\text{A.28})$$

or, in the vanishing  $\epsilon$  limit,

$$J_\epsilon^{(1)}(a) = -\frac{\pi a}{2} - \frac{a^2}{4} \left[ \frac{1}{\epsilon} - \ln 2\pi + \gamma - 1 \right] + \dots. \quad (\text{A.29})$$

In a similar manner, we have

$$J^{(2)}(a) = -\frac{1}{2} \int_0^\infty dz \left[ \frac{z^2}{\sqrt{z^2 + a^2}} - z \right] = \frac{a^2}{4} \left[ \frac{1}{\epsilon} - \ln \frac{a}{2} \right], \quad (\text{A.30})$$

with the help of

$$\int_0^\infty \frac{dz z^{2-\epsilon}}{\sqrt{z^2 + a^2}} = \frac{a^{2-\epsilon}}{2\sqrt{\pi}} \Gamma\left(\frac{3-\epsilon}{2}\right) \Gamma\left(-1 + \frac{\epsilon}{2}\right) = \frac{1}{\epsilon} - \ln \frac{a}{2} + O(\epsilon), \quad (\text{A.31})$$

and discarding the  $a$ -independent term. Adding up (A.29) and (A.30) then leads the final result

$$\begin{aligned} J(a) &= \int_0^\infty \frac{dz \, z^2}{\sqrt{z^2 + a^2}} \left( e^{\sqrt{z^2 + a^2}} - 1 \right)^{-1} \\ &= \frac{\pi^2}{6} - \frac{\pi a}{2} - \frac{a^2}{4} \left[ \ln \frac{a}{4\pi} + \gamma - 1 \right] + \dots \end{aligned} \quad (\text{A.32})$$

Finally, to evaluate (A.21), we observe that

$$\frac{\partial J(a, b)}{\partial b} = b \, I(a, b). \quad (\text{A.33})$$

Integrating over  $b$  with the help of (A.15) and imposing the boundary condition (A.32) gives

$$\begin{aligned} J(a, b) &= \frac{\pi^2}{6} - \frac{\pi}{2} \sqrt{a^2 - b^2} - \frac{1}{4} (a^2 - b^2) \left[ \ln \frac{a}{4\pi} + \gamma + \tanh^{-1} \frac{b}{a} \right] \\ &\quad + \sqrt{a^2 - b^2} \sin^{-1} \frac{b}{a} + \frac{a^2}{4} - b - \frac{b^3}{36} + \dots \end{aligned} \quad (\text{A.34})$$

The above expression can be checked by taking the limit  $a = b$  where (A.21) can be simplified using [22]:

$$\int_0^r \frac{du \, u^\ell}{e^u - 1} = r^\ell \left[ \frac{1}{\ell} - \frac{r}{2(\ell + 1)} + \sum_{n=1}^\infty \frac{B_{2n} r^{2n}}{(2n + \ell)(2n)!} \right] \quad (\ell \geq 1), \quad (\text{A.35})$$

and leads to

$$\begin{aligned} J^*(a) &= \int_0^\infty dz \, \frac{z}{e^{\sqrt{z^2 + a^2}} - 1} = \int_a^\infty du \, \frac{u}{e^u - 1} \\ &= \frac{\pi^2}{6} - a \left[ 1 - \frac{a}{4} + \sum_{n=1}^\infty \frac{B_{2n} a^{2n}}{(2n + 1)!} \right]. \end{aligned} \quad (\text{A.36})$$

With  $B_2 = 1/6$ , ones finds an amazing agreement between  $J^*(a)$  and  $J(a, b = a)$  in (A.34). Similarly, for  $b \rightarrow a \rightarrow \infty$ , we have

$$J(a, b \rightarrow \infty) = e^{-a} \left[ 1 + \frac{a}{2} + \frac{b^2}{2a} + \dots \right]. \quad (\text{A.37})$$

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## FIGURE CAPTIONS

Fig. 1. Diagrammatic representation of loop resummation. Contributions from all non-overlapping graphs are included in our physical loop.

Fig. 2. Flow pattern of the blocked potential for  $T = 0$  with  $\tilde{\mu}_R^2 = 10^{-4}$ ,  $\tilde{\lambda}_R = 0.1$  and  $\Lambda = 10$ . Solid and dashed lines represent the RG and IMA results, respectively.

Fig. 3. Flow pattern of the blocked potential for  $T = 5$  with  $\tilde{\mu}_R^2 = 10^{-4}$ ,  $\tilde{\lambda}_R = 0.1$  and  $\Lambda = 10$ . Solid and dashed lines represent the RG and IMA results, respectively.

Fig. 4. Evolution of  $\mu_{\beta,k}^2$  and  $\lambda_{\beta,k}$  as a function of  $k$  for various values of  $T$  using  $\tilde{\mu}_R^2 = 10^{-4}$ ,  $\tilde{\lambda}_R = 0.1$  and  $\Lambda = 20$ .

Fig. 5. Comparison of the flow of the mass parameter at  $T = 0$  for different RG prescriptions using  $\tilde{\mu}_R^2 = 10^{-4}$ ,  $\tilde{\lambda}_R = 0.1$ ,  $\tilde{\Lambda} = 30$ , and  $\Lambda = \tilde{\Lambda}/\sqrt{2}$ .

Fig. 6. Comparison of the flow of the coupling constant at  $T = 0$  for different RG prescriptions using  $\tilde{\mu}_R^2 = 10^{-4}$ ,  $\tilde{\lambda}_R = 0.1$ ,  $\tilde{\Lambda} = 30$ , and  $\Lambda = \tilde{\Lambda}/\sqrt{2}$ .

Fig. 7. Limitation on dimensional reduction at high temperature. Notice the gap between the mass parameters generated from the dimensionally reduced and the full RG prescriptions.

Fig. 8. Temperature dependence for  $\mu_\beta^2$  and  $\lambda_\beta$  at  $k = 0$ .

Fig. 9. Comparison of the critical temperature  $T_c$  obtained by RG with  $\tilde{T}_c$ . Notice that a higher value is predicted for RG.

Fig. 10. Temperature dependence of various quantities in the symmetry broken phase.